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Solutions to the Helmholtz equation for TE-guided waves in a three-layer structure with Kerr-type nonlinearity

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Abstract
We study certain solutions (TE-polarized electromagnetic waves) of the Helmholtz equation on the line describing waves propagating in a nonlinear three-layer structure consisting of a film surrounded by semi-infinite media. All three media are assumed to be lossless, nonmagnetic, isotropic and exhibiting a local Kerr-type dielectric nonlinearity. The linear component of the permittivity is modelled by a continuous real-valued function of the transverse coordinate. We show that the solution of the Helmholtz equation in the form of a TE-polarized electromagnetic wave exists and can be obtained by iterating the equivalent Volterra equation. The associated dispersion equation has a simple root (if the semi-infinite media are linear and if the nonlinearity parameter of the film is sufficiently small) that uniquely determines this solution.

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1. Introduction

Planar optical waveguides formed by one or several layers of Kerr-type nonlinear media find broad applications in various optical systems of signal processing [1, 2]. Most of the publications consider the permittivity $\epsilon_f = \bar{\epsilon}_f + a_f|\vec{E}|^2$ of the central layer (film) with $\bar{\epsilon}_f$ being constant [1, 2]. In this paper we assume that $\bar{\epsilon}_f = \bar{\epsilon}_f(z)$ is a real-valued continuous function of the transverse coordinate z. Even if we do not take into account the physical applications ([1], pp 257–279) of such a formulation of the problem, the case $\bar{\epsilon}_f = \bar{\epsilon}_f(z)$ is of

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independent interest from the viewpoint of mathematical physics. In the linear case ($a_f = 0$), the study of the wave propagation in a layered dielectric is reduced in [3] to a self-adjoint boundary eigenvalue problem. In this paper, we consider waves propagating in a three-layer structure formed by the film situated between two semi-infinite nonlinear media (substrate s and cladding c) with the permittivities $\epsilon_\nu = \bar{\epsilon}_\nu + a_\nu|\vec{E}|^2$ ($\nu = s, c$), where ϵ_ν and a_ν are real and $\bar{\epsilon}_\nu$ do not depend on z. We will show that in this case, under certain assumptions, the solution to the Helmholtz equation (3) in the form of a TE-polarized electromagnetic wave exists and can be obtained by iterating the equivalent Volterra equation. We also obtain a dispersion equation for determination of longitudinal wavenumbers of eigenwaves propagating in the nonlinear structure under study; all quantities that enter the dispersion equation are determined in terms of solutions to the Volterra equation.

2. Formulation

In [4] we investigated the propagation of TE-waves in a nonlinear lossless isotropic three-layer dielectric waveguide with constant ϵ_ν. We obtained the conditions for the existence of certain solutions ([4], section 3). In this paper we continue the analysis of [5] by considering a structure with the permittivity

$$k_0^2\epsilon(z) = \begin{cases} \bar{\epsilon}_s + a_s|\vec{E}|^2 & z < 0 \\ \bar{\epsilon}_f(z) + a_f|\vec{E}|^2 & 0 \leqslant z \leqslant d \\ \bar{\epsilon}_c + a_c|\vec{E}|^2 & z > d. \end{cases} \tag{1}$$

Here, $\vec{E}$ is the electric field in the layers and $k_0 = \frac{\omega}{c}$ is the wavenumber of the free space. The geometry of the problem corresponds to the case considered in [4]. We will look for nontrivial (particular) solutions to the Maxwell equations, as in [5], in the form of TE-waves

$$\vec{E} = \vec{e}_y\phi(z)\,\mathrm{e}^{\mathrm{i}(k_0\gamma x - \omega t)} \tag{2}$$

where $\vec{e}_y = (0, 1, 0)^T$ is the unit vector of the axis Oy, and the effective longitudinal wavenumber $n = k_0\gamma$ in (2) is the spectral parameter of the problem. From representation (2), it follows (as a result of the substitution into the Maxwell equations, as in [5]) that $\phi(z)$ satisfies the Helmholtz equation

$$\phi''(z) - \left(n^2 - k_0^2\epsilon(z)\right)\phi(z) = 0. \tag{3}$$

The desired solution of equation (3) must be twice continuously differentiable in each layer and continuously differentiable (due to the continuity of the tangential components of the electric and magnetic fields). In addition, the solution must satisfy $\phi(z) \to 0$ for $|z| \to \infty$.

Using equations (16)–(19) derived in [4], one can show that in the substrate ($\nu = s$) and cladding ($\nu = c$), the solution of (3) is given by

$$\phi_s = \frac{q_s E_0}{q_s\cosh(q_s z) - \sqrt{q_s^2 - \frac{a_s}{2}E_0^2}\sinh(q_s z)} \tag{4}$$

$$\phi_c = \frac{q_c E_d}{q_c\cosh[q_c(z-d)] + \sqrt{q_c^2 - \frac{a_c}{2}E_d^2}\sinh[q_c(z-d)]} \tag{5}$$

where $q_\nu^2 = n^2 - \bar{\epsilon}_\nu$, $q_s^2 \geqslant \frac{a_s}{2}E_0^2, q_c^2 \geqslant \frac{a_c}{2}E_d^2$, q_ν ($\nu = s, c$) are either positive or purely imaginary with a positive imaginary part and $E_0 > 0$ and E_d are real constants. Functions ϕ_s and ϕ_c satisfy the boundary conditions at the points $z = 0$ and $z = d$, respectively:

$$\phi_s(0) = E_0 = \phi_f(0) \qquad \left.\frac{\mathrm{d}\phi_s}{\mathrm{d}z}\right|_{z=0} = E_0\sqrt{q_s^2 - \frac{a_s}{2}E_0^2} = \left.\frac{\mathrm{d}\phi_f}{\mathrm{d}z}\right|_{z=0} \tag{6}$$

$$\phi_c(d) = E_d = \phi_f(d) \qquad \left.\frac{\mathrm{d}\phi_c}{\mathrm{d}z}\right|_{z=d} = -E_d\sqrt{q_c^2 - \frac{a_c}{2}E_d^2} = \left.\frac{\mathrm{d}\phi_f}{\mathrm{d}z}\right|_{z=d}. \tag{7}$$

Note that (4), (5) and (6), (7) can be applied for arbitrary sign of parameters a_ν, $\nu = s, c$, and remain valid in the linear case ($a_\nu = 0$, $\nu = s, c$).

For equation (3) considered in the interval $0 < z < d$ it is known [6] that the Volterra equation

$$\phi_f(z) = \phi_0(z) - \int_0^z K(z-t)\big[\delta(t) + a_f\phi_f^2(t)\big]\phi_f(t)\,\mathrm{d}t \qquad \delta(z) = \bar{\epsilon}_f(z) - \bar{\epsilon}_f^0 \tag{8}$$

with the kernel

$$K(u) = \frac{\sinh(q_f u)}{q_f} \tag{9}$$

is equivalent to (3) and function $\phi_f(z)$ satisfies conditions (6). In (9), q_f is defined by $q_f = \sqrt{n^2 - \bar{\epsilon}_f^0}$, the constant $\bar{\epsilon}_f^0 \geqslant 1$, and

$$\phi_0(z) = E_0\left[\cosh(q_f z) + \frac{\sqrt{q_s^2 - \frac{a_s}{2}E_0^2}}{q_f}\sinh(q_f z)\right] \tag{10}$$

is a solution of (3) when $\epsilon(z) = \bar{\epsilon}_f^0$ and satisfies boundary conditions (6).

If $a_f = 0$, solution $\phi_f(z)$ is represented by a uniformly convergent series

$$\phi_f(z) = \sum_{j=0}^{\infty}\phi_j(z) \qquad \phi_j(z) = -\int_0^z K(z-t)\delta(t)\phi_{j-1}(t)\,\mathrm{d}t \qquad j = 1, 2, \ldots \tag{11}$$

where $\delta(z)$ is assumed to be continuous. In the nonlinear case ($a_f \neq 0$), solution $\phi_f(z)$ has the form of a limit function of a uniformly convergent functional sequence $\phi_j(z)$ ($j = 1, 2, \ldots$):

$$\phi_j(z) \to \phi_f(z) \qquad \phi_j(z) = \phi_0(z) - \int_0^z K(z-t)\left[\delta(t) + a_f\phi_{j-1}^2(t)\right]\phi_{j-1}(t)\,\mathrm{d}t \qquad j = 1, 2, \ldots. \tag{12}$$

Statements (11) and (12) are proved in the following section.

In [5], we have formulated the sufficient condition for a uniform convergence of (12) in the form

$$\max_{0\leqslant z\leqslant d}\left\{\int_0^z |K(z-t)|\,\mathrm{d}t[|\delta(z)| + |a_f||\phi_0(z)|^2]\right\} < \frac{4}{27}. \tag{13}$$

We give a proof of this condition in the next section.

Using conditions (7) and evaluating (8), (9), (10) and (12), we obtain the dispersion equation [5]

$$F(n, \xi) = 0 \tag{14}$$

where $\xi \in A$ indicates an element of the parameter set $A = \left\{\bar{\epsilon}_s, \bar{\epsilon}_c, \bar{\epsilon}_f^0, a_\nu, E_0, d\right\}$, and $F(n, \xi)$ is given by

$$F(n, \xi) \equiv \int_0^d K_1(d-t)\left[\delta(t) + a_f\phi_f^2(t)\right]\phi_f(t)\,\mathrm{d}t - \left.\frac{\mathrm{d}\phi_0}{\mathrm{d}z}\right|_{z=d} - E_d\sqrt{q_c^2 - \frac{a_c}{2}E_d^2} \tag{15}$$

with $E_d = \phi_f(d)$ according to equation (8) and

$$K_1(u) = \cosh(q_f u). \tag{16}$$

If a solution to equation (8) (or (3)) is determined using relationships (12) and (13) (or (11) in the linear case), then, the dispersion equation (14) is solved for the given function $\delta(z)$ with respect to n or one or several (all) parameters of the set A. Equation (14) implicitly defines $n = n(\xi)$.

In section 4 we prove, under certain assumptions, the existence of the implicit function $n(\xi)$ for the case when all parameters are fixed except for one, $\xi = a_f$, by applying the implicit function theorem.

3. Convergence of iterations

3.1. Linear case

First, the case $a_s = a_c = a_f = 0$ is considered. We prove that series (11) converges uniformly on $[0, d]$ to a continuous function $\phi_f(z)$ and estimate the norm $\|\phi_f\| = \max_{0\leqslant z\leqslant d} |\phi_f(z)|$. The function $\phi_f(z)$ satisfies (8) with $a_f = 0$ and $\phi_0(z)$ given by (10) with $a_s = 0$. Denoting

$$k(z,t) = \delta(t)K(z-t)$$

we estimate

$$|\phi_j(z)| \leqslant \|\phi_0\| \cdot \|k\|^j \frac{z^j}{j!} \tag{17}$$

where

$$\|k\| = \max_{0\leqslant z,t\leqslant d} |k(z,t)| \tag{18}$$

and

$$\phi_j(z) = \int_0^z k(z,t)\phi_{j-1}(t)\,\mathrm{d}t \qquad j = 1, 2, \ldots. \tag{19}$$

For $j = 1$ (19) implies

$$|\phi_1(z)| \leqslant \|\phi_0\| \cdot \|k\| \cdot z.$$

Assuming that (17) is valid, we obtain the inequality

$$|\phi_{j+1}(z)| \leqslant \|\phi_0\| \cdot \|k\|^{j+1} \cdot \int_0^z \frac{t^j}{j!}\,\mathrm{d}t \tag{20}$$

which yields the required estimate (17). In this case, solution $\phi_f(z)$ to equation (3) is represented by series (11) which converges uniformly on $[0, d]$ and, according to (17),

$$|\phi_f(z)| \leqslant \|\phi_0\| \cdot \sum_{j=0}^{\infty} \|k\|^j \frac{z^j}{j!} = \|\phi_0\|\,\mathrm{e}^{z\|k\|} \tag{21}$$

which yields the estimate for $\phi_f(z)$ in the space $C([0, d])$

$$\|\phi_f\| \leqslant \|\phi_0\|\,\mathrm{e}^{d\|k\|}. \tag{22}$$

Combining (11) and (17) we obtain

$$|\phi_f(z)| \leqslant \|\phi_0\|\,\mathrm{e}^{\|\delta\| p(z)} \qquad p(z) = \frac{\cosh(q_f z) - 1}{q_f^2} \qquad z \in [0, d]. \tag{23}$$

It should be noted that, if $\delta(z)$ is assumed to be a bounded measurable function, then the norm of k in $C([0, d])$ (18) should be replaced by the norm

$$\|k\|_{L^\infty} = \operatorname{esssup}_{0\leqslant z,t\leqslant d}|k(z,t)|.$$

3.2. Nonlinear case

Considering the case $a_s \neq 0$, $a_c \neq 0$ and $a_f \neq 0$ and assuming that $\delta(z)$ in (8) is a continuous function we solve the Volterra equation (8) using the sequence of functions (12).

Denoting

$$c_0 = \max_{0\leqslant z\leqslant d}[|\delta(z)| + |a_f||\phi_0(z)|^2] \max_{0\leqslant z\leqslant d}\int_0^z |K(z-t)|\,\mathrm{d}t \tag{24}$$

and assuming $c_0 < 1$ we prove the following.

Lemma 1. *For every* $j = 1, 2, \ldots,$ *the estimates*

$$\|\phi_j\| \leqslant a_j\|\phi_0\| \tag{25}$$

hold with

$$a_j = 1 + c_0 a_{j-1}^3 \qquad a_0 = 1. \tag{26}$$

Proof. We prove (25) by induction. From (12) and (24), we have, setting $j = 1$,

$$|\phi_1(z)| \leqslant \|\phi_0\|(1 + c_0)$$

which proves (25) in this case. Now, assume that (25) holds for any $j \geqslant 1$ and prove that it is valid for $j + 1$. From (12), (24) and (25), it follows that

$$\begin{aligned}|\phi_{j+1}(z)| &\leqslant \|\phi_0\| + \int_0^z |K(z-t)|\left[\|\delta\| + |a_f|\|\phi_0\|^2 a_j^2\right]\|\phi_0\|a_j\,\mathrm{d}t\\ &\leqslant \|\phi_0\|\cdot\left(1 + c_0 a_j^3\right) = \|\phi_0\|a_{j+1}.\end{aligned} \tag{27}$$

which proves the lemma. □

Lemma 2. *For every* $j = 1, 2, \ldots,$ *the estimates*

$$\|\phi_{j+1} - \phi_j\| \leqslant c_0\|\phi_0\|\cdot a_j^3 \tag{28}$$

hold, where a_j *are determined from (26).*

Proof. We prove (28) by induction. For $j = 0$, (28) is satisfied. From (12), we obtain

$$|\phi_{j+2}(z) - \phi_{j+1}(z)| \leqslant \int_0^z |K(z-t)||\phi_{j+1}(t) - \phi_j(t)|\left[|\delta(t)| + |a_f|\left(\phi_{j+1}^2 + \phi_{j+1}\phi_j + \phi_j^2\right)\right]\mathrm{d}t. \tag{29}$$

If (28) holds for every $j \geqslant 0$, we use (24) and lemma 1 to find the upper bound of the right-hand side of (29),

$$c_0\|\phi_0\|c_0 a_j^3\left(a_{j+1}^2 + a_{j+1}a_j + a_j^2\right). \tag{30}$$

To complete the proof, it is sufficient to show that

$$c_0 a_j^3\left(a_{j+1}^2 + a_{j+1}a_j + a_j^2\right) \leqslant \left(1 + c_0 a_j^3\right)^3. \tag{31}$$

In order to prove this inequality we choose a c_0 that satisfies

$$c_0 \leqslant \frac{\alpha}{(1+\alpha)^3} \tag{32}$$

for a certain positive $\alpha \leqslant 1/2$. Then, taking into account that $a_{j+1} > a_j$, we obtain the upper bound $3c_0 a_j^3 a_{j+1}^2$ for the left-hand side of (31). Condition (32) yields $c_0 \leqslant 4/27$ (cf (13)) for $\alpha \leqslant 1/2$ and $a_j \leqslant 1 + \alpha$ for every $j = 0, 1, 2, \ldots$. This implies

$$2c_0 a_j^3 \leqslant 1 \tag{33}$$

and thus

$$3c_0a_j^3a_{j+1}^2 \leqslant a_{j+1}^3 = \left(1 + c_0a_j^3\right)^3. \tag{34}$$

The lemma is proved. □

Lemma 3. *If c_0 satisfies (32) for a certain positive $\alpha \leqslant 1/2$, then sequence (12) converges in $C([0, d])$ to a continuous function $\phi_f(z)$ that satisfies equation (8).*

Proof. We prove that $\{\phi_j(z)\}$, $j = 0, 1, 2, \ldots$, is a Cauchy sequence in $C([0, d])$. Indeed,

$$\|\phi_{j+p} - \phi_j\| \leqslant \|\phi_{j+p} - \phi_{j+p-1}\| + \cdots + \|\phi_{j+1} - \phi_j\| \tag{35}$$

holds. We estimate $\|\phi_{j+1} - \phi_j\|$ for $j \geqslant 1$. In what follows, we will assume that

$$\max_{0\leqslant t\leqslant z\leqslant d}\{[|\delta(z)| + |a_f||\phi_0(z)|^2] \cdot |K(z - t)|\} \leqslant 1. \tag{36}$$

This inequality is consistent with condition (13) and implies that the norm of function $\delta(z)$, the nonlinearity parameter a_f and the film thickness d must be taken sufficiently small. Using induction, we prove the inequality

$$|\phi_{j+1}(z) - \phi_j(z)| \leqslant \|\phi_0\|a_j^3\frac{(z)^{j+1}}{(j + 1)!}. \tag{37}$$

For $j = 0$ it follows from (12) and (36) that

$$|\phi_1(z) - \phi_0(z)| \leqslant z\|\phi_0\|$$

which coincides with (37) for $j = 0$ because $a_0 = 1$.

Assuming that (37) holds for every $j \geqslant 0$ we prove that it will be valid for the subsequent value $j + 1$. According to lemma 1 and (36), we obtain

$$\begin{aligned}|\phi_{j+2}(z) - \phi_{j+1}(z)| &\leqslant \int_0^z |K(z - t)| \left[|\delta(t)| + |a_f| \left(\phi_{j+1}^2 + \phi_{j+1}\phi_j + \phi_j^2\right)\right] \|\phi_0\|a_j^3\frac{t^{j+1}}{(j + 1)!}\, \mathrm{d}t \\ &\leqslant \|\phi_0\|a_j^3 \left(a_{j+1}^2 + a_{j+1}a_j + a_j^2\right) \frac{z^{j+2}}{(j + 2)!} \leqslant \frac{\left(1 + c_0a_j^3\right)^3}{c_0}\frac{z^{j+2}}{(j + 2)!}\|\phi_0\|. \end{aligned} \tag{38}$$

Using inequality (31) we estimate the right-hand side of (38) from above by

$$\|\phi_0\|\frac{a_{j+1}^3}{c_0}\frac{z^{j+2}}{(j + 2)!}$$

which completes the proof of (37).

Inequality (37) yields an upper bound for the right-hand side of (38):

$$\begin{aligned}&\frac{1}{c_0}\|\phi_0\|\left(\frac{a_j^3d^{j+1}}{(j + 1)!} + \cdots + \frac{a_{j+p-1}^3d^{j+p}}{(j + p)!}\right) \\ &\quad\leqslant \frac{d^{j+1}}{c_0}\frac{(1 + \alpha)^3}{(j + 1)!}\left(1 + \frac{d}{j + 2} + \frac{d^2}{(j + 2)(j + 3)} + \cdots\right)\|\phi_0\| \\ &\quad\leqslant \frac{\alpha}{c_0^2}\frac{d^{j+1}}{(j + 1)!}\frac{\|\phi_0\|}{1 - d/(j + 2)} \to 0 \qquad j \to \infty. \end{aligned} \tag{39}$$

Thus, lemma 3 is proved. □

It should be noted that the condition $c_0 \leqslant \frac{4}{27}$ imposes, together with (13), (32) and (36), restrictions on $\bar{\epsilon}_f(z)$ and $a_fE_0^2$ or the layer thickness d.

4. Existence

To prove the existence of solutions to (3) we first consider a permittivity $\epsilon(z)$ given by (1) with $a_s = a_c = a_f = 0$,

$$k_0^2\epsilon(z) = \begin{cases} \bar{\epsilon}_s & z < 0 \\ \epsilon_f(z) & 0 \leqslant z \leqslant d \\ \bar{\epsilon}_c & z > d \end{cases} \tag{40}$$

with real constants $\bar{\epsilon}_{c,s}$ and $\epsilon_f(z)$ being a real-valued function continuous on $[0, d]$. In this case, if $\bar{\epsilon}_s = \bar{\epsilon}_c$, the Helmholtz equation (3) is equivalent to

$$-\phi''(z) + q(z)\phi(z) = \lambda\phi \tag{41}$$

with

$$q(z) = \begin{cases} \bar{\epsilon}_c - \epsilon_f(z) & 0 \leqslant z \leqslant d \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

and $\lambda = \bar{\epsilon}_c - n^2$. As is known [7], equation (41) with a compactly supported potential $q(z)$ given by (42) has nontrivial solutions (eigenfunctions) at most finitely many negative eigenvalues λ of the multiplicity one (the number of eigenvalues may be equal to zero). Assume that, for a given continuous function $\epsilon_f(z)$, there exists at least one such eigenvalue $\lambda = \lambda^*$ (according to [8], this holds, for example, if $q(z)$ is not identically zero and $q(z) \leqslant 0$); this is equivalent to the assumption of the existence of a simple root $n = n^* = \sqrt{\bar{\epsilon}_c - \lambda^*}$ of the dispersion equation (14) for $a_s = a_c = a_f = 0$. Hence

$$\left.\frac{\partial F}{\partial n}\right|_{n=n^*} \neq 0 \tag{43}$$

holds (for fixed parameters from the set A).

If $\bar{\epsilon}_s \neq \bar{\epsilon}_c$, the potential $q(z)$ is not a compactly supported function and eigenvalues are at most finite, each having the finite multiplicity [9]. In what follows, we consider only the case of simple roots of the dispersion equation (14) (simple eigenvalues) supposing that (43) is valid.

To analyse the perturbation of roots of the dispersion equation (14) for $a_s = a_c = 0$ and $a_f \neq 0$ due to small variation of the parameter $\xi = a_f$ we assume that, for a given continuous function $\epsilon_f(z)$, there exists at least one simple root $n = n^*$ of the dispersion equation (14) for $a_s = a_c = a_f = 0$ (one negative eigenvalue $\lambda^* = \bar{\epsilon}_c - (n^*)^2$ of equation (41) of the multiplicity one) and consider (14) for fixed $\bar{\epsilon}_s, \bar{\epsilon}_c, \bar{\epsilon}_f^0, a_\nu, E_0, d$, in the vicinity of $n = n^*, \xi = 0$. Since $\frac{\partial F}{\partial n}(n, \xi)$ is continuous and

$$\begin{cases} F(n^*, 0) = 0 \\ \frac{\partial F}{\partial n}(n^*, 0) \neq 0 \end{cases} \tag{44}$$

the implicit function theorem [10] yields the unique existence of a continuous function $n(\xi)$ in a small vicinity of $n = n^*, \xi = 0$.

To sum up, we have proved the following.

Theorem. *Assume that the following conditions are satisfied:*

(i) *for a given continuous function $\epsilon_f(z)$ ($z \in [0, d]$) and positive constants $\bar{\epsilon}_c$ and $\bar{\epsilon}_s$ there exists at least one eigenvalue of equation (41);*

(ii) *the function $\delta(z) = \epsilon_f(z) - \bar{\epsilon}_f^0$ and a positive constant $\bar{\epsilon}_f^0$ are such that the number c_0 defined by (24) satisfies the condition $c_0 < 4/27$;*

(iii) *inequality (36) holds.*

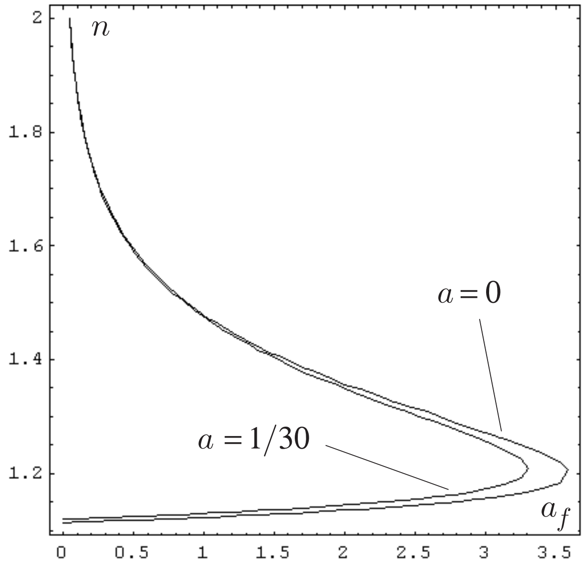

Figure 1. The relation between a_f and n for $a = \frac{1}{30}$ and 0.

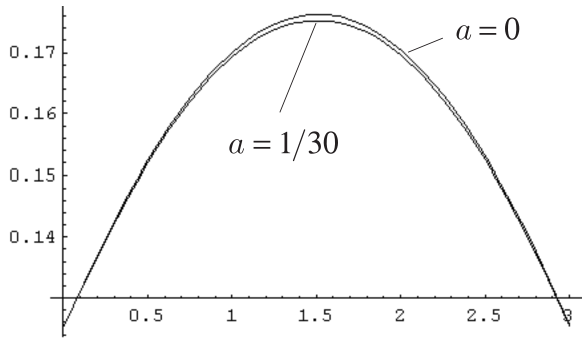

Figure 2. The field patterns inside the film for $a = \frac{1}{30}$ and 0.

Then there is a sufficiently small $a_f^* > 0$ *such that for all* $\xi = a_f \in (0, a_f^*)$*, there exists a solution to the Helmholtz equation (3) according to (4), (5) and (12). The corresponding root* $n = n(\xi)$ *of the dispersion equation (14) exists and is simple.*

One can determine the range of variation of the internal layer thickness d and nonlinearity parameter $\xi = a_f$ (in the form of a nonempty domain D in the (d, ξ)-plane) that satisfy conditions (13), (32) and (36). Indeed, assume that for fixed $\bar{\epsilon}_\nu, a_\nu$ $(\nu = s, c)$, and $\bar{\epsilon}_f^0$ and a (continuous) function $\epsilon_f(z)$, and $\xi \in (0, a_f^*)$, the corresponding root of equation (14) $n = n(\xi) \in (n_1, n_2)$, where n_1 and n_2 are positive. Then we can use (9) and write the estimates

$$\max_{0\leqslant t\leqslant z\leqslant d} |K(z-t)| \leqslant M_0(d) = \frac{\sinh(Q_f d)}{Q_f} \qquad Q_f = \sqrt{n_2^2 - \bar{\epsilon}_f^0} > 0 \tag{45}$$

where $M_0(d) \to 0$ as $d \to 0$, and also

$$\max_{0\leqslant z\leqslant d} \int_0^z |K(z-t)|\, \mathrm{d}t \leqslant \mathrm{d}M_0(d) \tag{46}$$

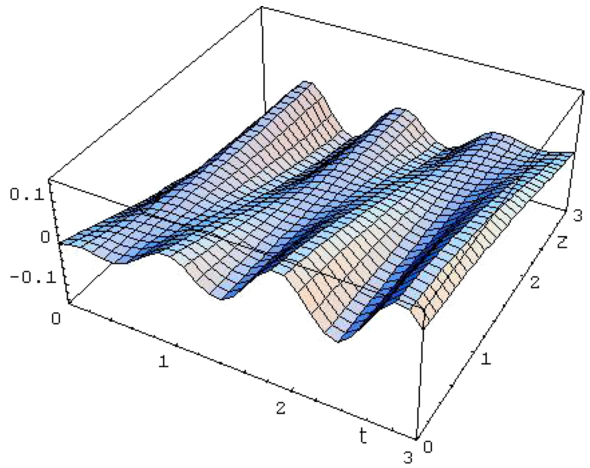

Figure 3. The left-hand side of condition (36) for $a = 1/30$.

$$\|\phi_0\| \leqslant M_1(d) \qquad M_1(d) = E_0\left[\cosh(Q_f d) + M_0(d)\sqrt{Q_s^2 - \frac{a_s}{2}E_0^2}\right] \qquad Q_s^2 = n_2^2 - \bar{\epsilon}_s. \tag{47}$$

Now we can use (47) to estimate from above the quantity c_0 given by (24) and write the inequalities

$$\mathrm{d}M_0(d)\left[\delta_0 + M_1^2(d)|\xi|\right] < \tfrac{4}{27} \tag{48}$$

which yields condition (32) with $\alpha = 1/2$ and

$$M_0(d)\left[\delta_0 + M_1^2(d)|\xi|\right] \leqslant 1 \tag{49}$$

which yields condition (36); here $\delta_0 = \|\delta(z)\|$. Obviously, for every ξ, there exists a (sufficiently small) d_0 such that the system of inequalities (48) and (49) is satisfied when $d \in [0, d_0]$. The solution to (48) and (49) specifies a nonempty subdomain $D' \subset D$ where conditions (13), (32) and (36) hold.

A simplified analysis of the solution to the system of inequalities (48) and (49) is given in the appendix.

5. A numerical example

To illustrate the above analysis we assume a periodic dependence of $\bar{\epsilon}_f(z)$ so that $\delta(z) = a\cos^2 bz/d$ and compare the results with those of $\bar{\epsilon}_f(z) = \bar{\epsilon}_f^0 = \text{const}$.

Choosing the parameters $\epsilon_s = \epsilon_c = 1, \bar{\epsilon}_f^0 = 1.5, d = 3, E_0 = \frac{1}{8}$ and $b = 10$ and evaluating the dispersion equation (14) with the first iteration of (8) we obtain a relation between a_f and n for $a = \frac{1}{30}$ and $a = 0$ (figure 1). The corresponding field patterns inside the film are shown in figure 2.

The difference between the cases $\delta = 0$ and $\delta \neq 0$ are small because the conditions $c_0 < 1$ and (36) had to be satisfied for $a_f = 0.2$. Figures 3 and 5 show the left-hand side of condition (36) for $a = 1/30$ and 0, respectively, both plotted with respect to z and t. Figures 4 and 6 show c_0 calculated according to (24) for the same a. One can see that the condition $c_0 < 1$

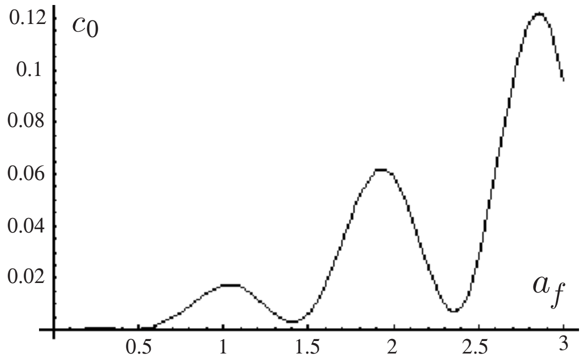

Figure 4. c_0 calculated according to (24) versus a_f for $a = 1/30$.

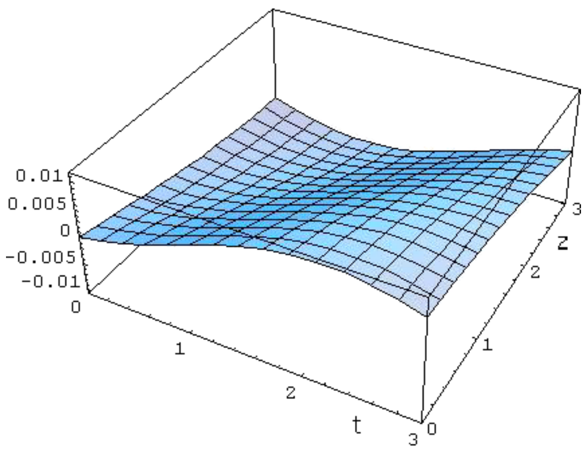

Figure 5. The left-hand side of condition (36) for $a = 0$.

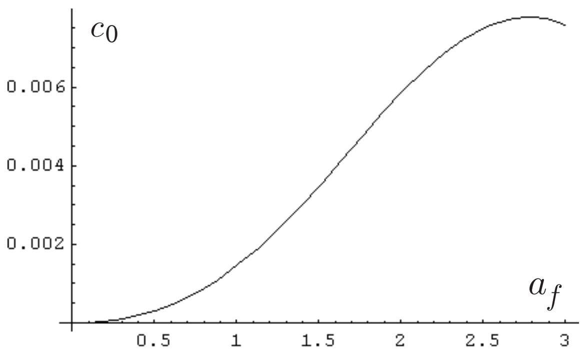

Figure 6. c_0 calculated according to (24) versus a_f for $a = 0$.

is satisfied for the chosen values of a. The field pattern in the entire three-layer structure is shown in figure 7 (note that we have chosen the plot range such that the asymptotic behaviour

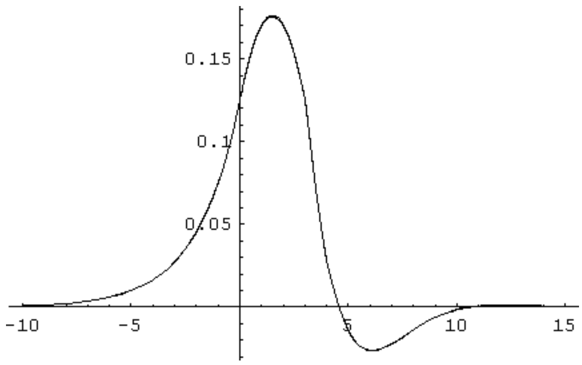

Figure 7. The field pattern in the entire three-layer structure.

of the fields ϕ_s and ϕ_c becomes obvious, which implies that the different field patterns in figure 2 are not visible).

6. Conclusion

The TE-polarized waves defined by (2) (with $\phi(z) \to 0$ as $|z| \to \infty$) supported by a lossless Kerr-like nonlinear three-layer structure modelled by a permittivity function $\epsilon(z)$ according to (1) have been investigated.

Firstly, we have proved (cf the theorem in section 4) that, subject to certain conditions, a solution of the Helmholtz equation (3) exists and can be presented in the form of a limit function $\phi_f(z)$ of a uniformly convergent functional sequence of iterations of (8).

Secondly, we have proved the unique existence of a continuous function $n(a_f)$ as a solution of the dispersion equation (14) (in a small vicinity of n^*, a_f according to (44)).

Thirdly, we have illustrated the mathematical results by a numerical example. With respect to possible/expected practical applications the analysis outlined above can be extended to more general functions $\epsilon(z)$ that model the permittivity in nonlinear photonic crystals [11], in particular, periodic dependences of the permittivity considered in section 5.

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Appendix

Assume that $d \leqslant 1$. Then (48) implies (49). Consider the solution to the inequality

$$M_0(d)\left[\delta_0 + M_1^2(d)|\xi|\right] < \tfrac{4}{27}. \tag{A1}$$

Denoting $v = \sinh(Q_f d)$, $\eta = |\xi|E_0^2$ and $\Theta = \sqrt{Q_s^2 - \frac{a_s}{2}E_0^2}$, we can rewrite (48) as an inequality in terms of the variables v and η,

$$G(v) \equiv \delta_0 v + \eta v\left(\frac{\Theta}{Q_f}v + \sqrt{1+v^2}\right) \leqslant \frac{4}{27}Q_f. \tag{A2}$$

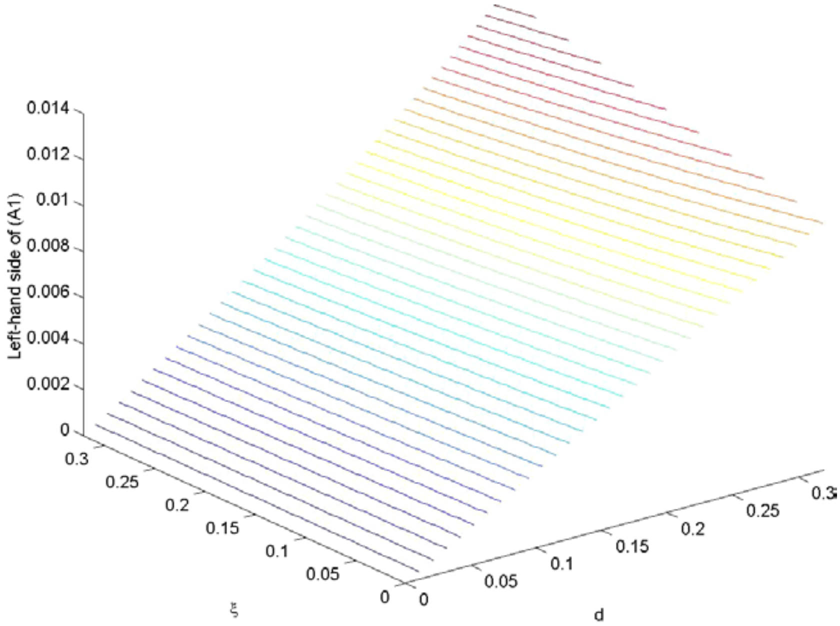

Figure 8. The left-hand side of inequality (A1) (vertical axis) plotted versus ξ and d; the parameters $\epsilon_s = \epsilon_c = 1$, $\bar{\epsilon}_f^0 = 1.5$, $E_0 = 1/8$ and $\delta_0 = 1/30$ correspond to the numerical example considered in section 5.

For every fixed ξ, $G(v)$ is a monotonically increasing positive function in every interval $[0, V]$, and $G(v) \to 0$ as $v \to 0$. Consequently, for every positive B, there exists a (positive) V_B such that the inequality $G(v) \leqslant B$ is satisfied when $v \in [0, V_B]$. In addition,

$$G(v) < G_0(v) = \eta\left(\frac{\Theta}{Q_f} + 1\right)v^2 + (\delta_0 + \eta)v$$

because $\sqrt{1+v^2} < 1 + v$ for positive v. Therefore, all v satisfying the inequality $G_0(v) \leqslant 4/27Q_f$,

$$0 < v < V_0 \qquad V_0 = V_0(\eta) = \frac{(\delta_0+\eta) + \sqrt{(\delta_0+\eta)^2 + 4(4/27Q_f)^2}}{2\left(\eta\left(\frac{\Theta}{Q_f}+1\right)\right)} \tag{A3}$$

also solve inequality (A2). Thus, (A1) is satisfied and conditions (13), (32) and (36) hold for all d satisfying

$$d \leqslant \min\{1, 1/Q_f \operatorname{arcsinh} V_0\}. \tag{A4}$$

A plot in figure 8 illustrates the behaviour of the left-hand side of (A1) for small d and ξ. One can see that condition (A1) is satisfied, for a chosen set of parameters (see the figure caption).

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